29/9/22 MATH4030 Tutorial.

1) Finished grading Assignment 1

- -20% late penalty for assignment 1.
- for Assignment 2 onwards will NOT accept any late submission

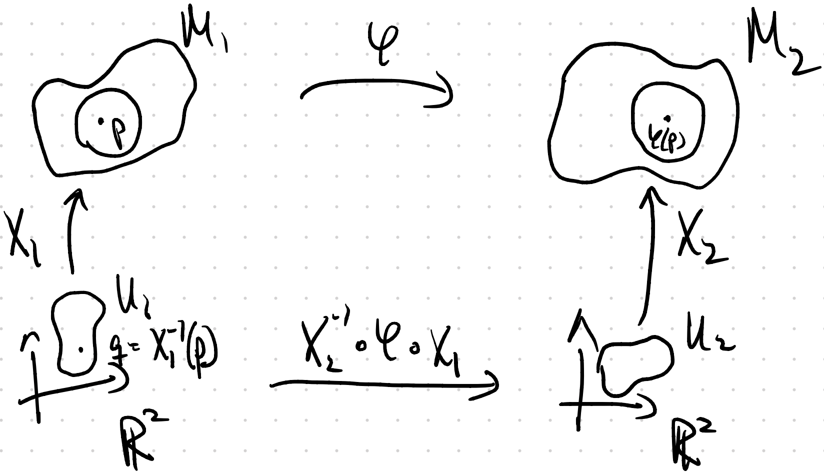
Last time, looked at notion of differentiable function $f : M \to \mathbb{R}$.

Now, notion of differentiable maps between surfaces.

<u>Def</u>: $\varphi : M_1 \to M_2$ is differentiable at $p \in M_1$, if given parametrizations

$$X_1 : U_1 \subset \mathbb{R}^2 \to M_1, \quad X_2 : U_2 \subset \mathbb{R}^2 \to M_2$$

with $p \in X_1(U_1)$, $\varphi(X_1(U_1)) \subset X_2(U_2)$, the map $X_2^{-1} \circ \varphi \circ X_1 : U_1 \to U_2$ is differentiable at $q = X_1^{-1}(p)$.



$\varphi : M_1 \to M_2$ is diffeomorphic if it is a differentiable map with differentiable inverse.

Recap Def: (First Fundamental Form): Given a regular surface M, $p \in M$, we have tangent plane T_pM. Then the first fundamental form g of M at p is the map $g_p : T_pM \times T_pM \to \mathbb{R}$ by

$$g_p(v,w) = \langle v, w \rangle.$$

where $\langle \, , \rangle$ is the standard inner / dot product on T_pM.

Let $X: U \to M$ be a parametrization, then the coefficients of g are given by

$$E = g(X_u, X_u) = \langle X_u, X_u \rangle$$
$$F = g(X_u, X_v) = \langle X_u, X_v \rangle$$
$$G = g(X_v, X_v) = \langle X_v, X_v \rangle.$$

Then for tangent vectors $V = a_1 X_u + b_1 X_v$, $W = a_2 X_u + b_2 X_v$, then

$$\begin{aligned} g(V,W) &= \langle a_1 X_u + b_1 X_v, a_2 X_u + b_2 X_v \rangle \\ &= \langle a_1 X_u, a_2 X_u + b_2 X_v \rangle + \langle b_1 X_v, a_2 X_u + b_2 X_v \rangle \\ &= \langle a_1 X_u, a_2 X_u \rangle + \langle a_1 X_u, b_2 X_v \rangle + \langle b_1 X_v, a_2 X_u \rangle \\ &\quad + \langle b_1 X_v, b_2 X_v \rangle \\ &= a_1 a_2 E + (a_1 b_2 + a_2 b_1) F + b_1 b_2 G. \end{aligned}$$

Recap Def (Lengths and Areas):

- $\alpha(t) = (x(t), y(t), z(t))$ smooth curve on M, $a \le t \le b$
  s.t. $\alpha(t) = X(u(t), v(t))$.

Then the length of α is given by

$$l = \int_a^b |\alpha'|(t)\, dt$$

$$= \int_a^b \left(g(\alpha'(t), \alpha'(t)) \right)^{\frac{1}{2}} dt$$

$$\alpha'(t) = \frac{d}{dt}\left(X(u(t), v(t)) \right) = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt}.$$

$$\begin{aligned} g(\alpha'(t), \alpha'(t)) &= \left\langle \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt}, \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt} \right\rangle \\ &= \left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial u} \right\rangle \left(\frac{du}{dt}\right)^2 + 2\left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \right\rangle \frac{du}{dt}\frac{dv}{dt} + \left\langle \frac{\partial X}{\partial v}, \frac{\partial X}{\partial v} \right\rangle \left(\frac{dv}{dt}\right)^2 \\ &= E(\alpha(t))\left(\frac{du}{dt}\right)^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))\left(\frac{dv}{dt}\right)^2 \end{aligned}$$

So $l = \int_a^b \left(E(\alpha(t))\left(\frac{du}{dt}\right)^2 + 2F(\alpha(t))\frac{du}{dt}\frac{dv}{dt} + G(\alpha(t))\left(\frac{dv}{dt}\right)^2 \right)^{\frac{1}{2}} dt$

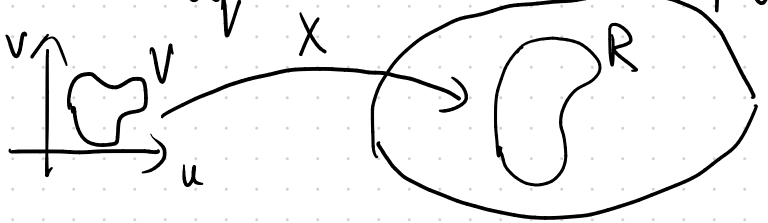
Writing (u^1, u^2) instead of (u, v) and $X_i = \frac{\partial X}{\partial u^i}$.

$$l = \int_a^b \left(\sum_{i,j=1}^{2} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt} \right)^{1/2} dt$$

where $g_{ij} = \langle X_i, X_j \rangle$ ← metric as in Riemannian geometry.

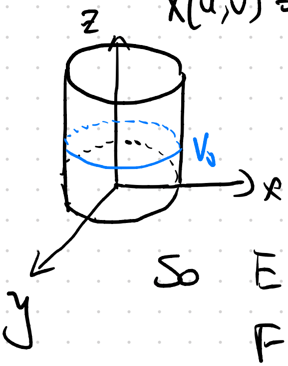
- Let $X: U \to M$ be a parametrization of a regular surface. Let R be a closed and bounded region in $X(U)$. Let $V = X^{-1}(R)$. The area of R is given by

$$A(R) = \iint_V |X_u \times X_v| \, du\, dv = \iint_V \sqrt{EG - F^2}$$



Ex1: Compute the first fundamental form of the cylinder given by the parametrization

$$X(u,v) = (\cos u, \sin u, v) \qquad U = \{(u,v) : 0 < u < 2\pi,\ -\infty < v < \infty\}.$$



$$X_u = (-\sin u, \cos u, 0)$$

$$X_v = (0, 0, 1)$$

So $E = \langle X_u, X_u \rangle = (-\sin u)^2 + \cos^2 u = 1$, ie

$F = \langle X_u, X_v \rangle = 0$

$G = \langle X_v, X_v \rangle = 1.$ $\qquad g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Note that these coincide with the coefficients of the first fundamental form of the plane.

Compute the length of the curve $u(t) = t$, $0 \le t \le 2\pi$, $v(t) = v_0$.

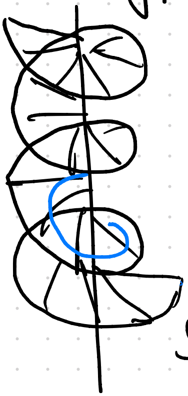
$\alpha(t) = (\cos t, \sin t, v_0)$, $\quad 0 \le t \le 2\pi$

$\alpha'(t) = (-\sin t, \cos t, 0).$

$$l = \int_0^{2\pi} (1 \cdot 1^2 + 2 \cdot 0 \cdot 1 \cdot 0 + 0 \cdot 1^2)^{\frac{1}{2}} dt = \int_0^{2\pi} dt = 2\pi$$

as to be expected since this is just the trace of a circle.

Ex 2: Compute the first fundamental form of the helicoid, the ruled surface obtained by taking a helix, and at each point, drawing a line parallel to the xy plane and intersecting the z-axis, given by the parametrization



$$X(u,v) = (v\cos u,\ v\sin u,\ u) \quad a>0,\ 0<u<2\pi,$$
$$-\infty<v<\infty.$$

$X_u = (-v\sin u,\ v\cos u,\ a)$

$X_v = (\cos u,\ \sin u,\ 0)$

So $E = \langle X_u, X_u\rangle = (-v\sin u)^2 + (v\cos u)^2 + 1^2$
$= v^2+1$

$F = \langle X_u, X_v\rangle = -v\sin u\cos u + v\cos u\sin u + 0 = 0.$

$G = \langle X_v, X_v\rangle = \cos^2 u + \sin^2 u + 0 = 1.$ $\quad g = \begin{pmatrix} 1 & 0 \\ 0 & v^2+1 \end{pmatrix}.$

Compute the length of the curve $u(t) = t$, $0 \le t \le 2\pi$, $v(t) = v_0$.

$\alpha(t) = (v_0\cos t,\ v_0\sin t,\ t)$. $\alpha'(t) = (-v_0\sin t,\ v_0\cos t,\ 1)$

$$l = \int_0^{2\pi} \left((v_0^2+1)\,1^2 + 2\cdot 0\cdot 1\cdot 0 + 1\cdot 0^2\right)^{1/2} dt$$

$$= \int_0^{2\pi} \sqrt{v_0^2+1}\, dt = 2\pi\sqrt{v_0^2+1}$$

Ex3: For previous two examples (cylinder, helicoid), compute the area of the region

$$R := \begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le 1. \end{cases}$$

Cylinder: $\sqrt{EG - F^2} = \sqrt{1 \cdot 1 - 0^2} = \sqrt{1} = 1.$

$$A(R) = \int_0^{2\pi} \int_0^1 dv\,du = \int_0^{2\pi} du = 2\pi$$

Helicoid: $\sqrt{EG - F^2} = \sqrt{(v^2+1) \cdot 1 - 0^2} = \sqrt{v^2+1}.$

$$A(R) = \int_0^{2\pi} \int_0^1 \sqrt{v^2+1}\, dv\, du = \int_0^{2\pi} \frac{1}{2}\left(\sqrt{2} + \sinh^{-1}(1)\right) du$$

$$= \pi \underbrace{\left(\sqrt{2} + \sinh^{-1}(1)\right)}_{\simeq 2.3}$$